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On Drinfeld's realization of quantum affine algebras

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We show how to obtain Drinfeld's realization of quantum nontwisted affine algebras from the quantized Cartan–Weyl basis. The formulae for comultiplications in this realization are discussed.

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1. Introduction

Quantum affine algebras and their representations are intensively studied now by physicists and mathematicians. These algebras are naturally connected with trigonometric solutions of the Yang–Baxter equation [6]. They appear also as symmetries of quantum deformations of integrable systems [5].

Quantum affine algebras are usually defined as Hopf algebras with Chevalley generators and with q-deformed standard relations. Unfortunately, this form is not convenient for some applications, like studying finite-dimensional representations and developing q-vertex operator calculus. Different approaches of defining quantum affine algebras as quantizations of current algebras were presented in the works [1,3,4,11].

The main goal of this lecture is to explain the mysterious formulae of ref. [3] from the point of view of algebraic calculations with Cartan–Weyl generators. We explain here how to obtain "quantized current generators" and commutation relations between them from the construction of Cartan–Weyl generators [7,8] for quantum affine algebras. The basic concepts in our approach are the notions of normal ordering of the root system and q-commutator of the root vectors. By use of the universal R-matrix which was written down explicitly in ref. [7] we describe the formulae for comultiplication in Drinfeld's realization of quantum affine algebras.

2. Notations

Let $\widehat{g}$ be a nontwisted affine Lie algebra with symmetrizable Cartan matrix $A = (a_{ij})$ ($A^{\rm sym} = (a_{ij}^{\rm sym})$ is the corresponding symmetrical matrix) and let $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ be a system of simple roots for $\widehat{g}$. We assume that the roots $\Pi_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ generate the system $\Delta_+(g)$ of positive roots of the corresponding finite-dimensional Lie algebra g. The quantum deformation $U_q(\widehat{g})$ is a unital algebra with generators $e_{\pm\alpha_i}$, $k_{\alpha_i}^{\pm 1} = q^{\pm h_{\alpha_i}}$ $(i = 1, 2, \ldots, r)$, and the defining relations

$$[k_{\alpha_i}^{\pm 1}, k_{\alpha_j}^{\pm 1}] = 0, \quad k_{\alpha_i} e_{\pm\alpha_j} = q^{\pm(\alpha_i, \alpha_j)} e_{\pm\alpha_j} k_{\alpha_i}, \tag{1}$$

$$[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}}, \tag{2}$$

$$(\mathrm{ad}_{q'}\, e_{\pm\alpha_i})^{1-a_{ij}} e_{\pm\alpha_j} = 0 \quad \text{for } i \neq j,\ q' = q, q^{-1}, \tag{3}$$

where $(\mathrm{ad}_q\, e_\alpha) e_\beta$ is a q-commutator:

$$(\mathrm{ad}_q\, e_\alpha) e_\beta \equiv [e_\alpha, e_\beta]_q = e_\alpha e_\beta - q^{(\alpha,\beta)} e_\beta e_\alpha \tag{4}$$

and (α, β) is a scalar product of the roots α and β: $(\alpha_i, \alpha_j) = a_{ij}^{\rm sym}$. We define a comultiplication in $U_q(\widehat{g})$ by the formulae

$$\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i}, \tag{5}$$

$$\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i}^{-1} \otimes e_{\alpha_i}\,, \quad \Delta(e_{-\alpha_i}) = e_{-\alpha_i} \otimes k_{\alpha_i} + 1 \otimes e_{-\alpha_i} \tag{6}$$

We denote by a symbol $(^*)$ an anti-involution in $U_q(\widehat{g})$, defined as $(k_{\alpha_i})^* = k_{\alpha_i}^{-1}$, $(e_{\pm\alpha_i})^* = e_{\mp\alpha_i}$, $(q)^* = q^{-1}$. We also use the standard notations $U_q(\widehat{b}_+)$ and $U_q(\widehat{b}_-)$ for the Borel subalgebras, generated by $k_{\alpha_i}^{\pm 1}$, e_{α_i} and $k_{\alpha_i}^{\pm 1}$, $e_{-\alpha_i}$ correspondingly. We also write

$$\exp_q(x) := 1 + x + \frac{x^2}{(2)_q!} + \ldots + \frac{x^n}{(n)_q!} + \ldots = \sum_{n \geq 0} \frac{x^n}{(n)_q}, \tag{7}$$

$$(a)_q := \frac{q^a - 1}{q - 1}, \quad [a]_q := \frac{q^a - q^{-a}}{q - q^{-1}}, \quad q_\alpha := q^{-(\alpha,\alpha)}. \tag{8}$$

3. Cartan–Weyl basis for $U_q(\widehat{g})$

Let Δ_+ be the system of all positive roots with respect to Π. It turns out that a procedure of the construction of the quantum Cartan–Weyl basis has to be in agreement with the choice of normal ordering in the reduced root system Δ_+. Recall the definition of normal order in Δ_+ [2,12].

Definition 3.1. We say that the system Δ_+ is in normal ordering if its roots are written in the following way: (i) all multiple roots follow each other in an arbitrary order; (ii) each non-simple root $\alpha + \beta \in \Delta_+$, where $\alpha \neq \lambda\beta$ has to be written between α and β.

Fix some normal ordering in $\Delta_+ \ (\widehat{g}) := \Delta_+$, satisfying an additional condition:

$$\alpha_i + n\delta < k\delta < (\delta - \alpha_j) + l\delta \tag{9}$$

for any simple roots $\alpha_i, \alpha_j \in \Delta_+ \ (g)$, $k, l, n \geq 0$. Here δ is a minimal positive imaginary root. Apply the following inductive procedure for the construction of the real root vectors $e_\gamma, e_{-\gamma}, ,\gamma \in \Delta_+ \ (\widehat{g})$, starting from the simple root vectors of $\Delta_+ \ (\widehat{g})$.

Let $\gamma \in \Delta_+ \ (\widehat{g})$ be a real root and $\alpha, \ldots, \gamma, \ldots, \beta$ be a minimal subset restricting γ ($\gamma = \alpha + \beta$). Then we set

$$e_\gamma := [e_\alpha, e_\beta]_q, \quad e_{-\gamma} := [e_{-\beta}, e_{-\alpha}]_{q^{-1}} \tag{10}$$

if $e_{\pm\alpha}$ and $e_{\pm\beta}$ have already been constructed.

When we get the imaginary root δ , we stop for a moment and use the following formulae:

$$e_\delta^{(i)} = \varepsilon_1(\alpha_i)[(\alpha_i, \alpha_i)]_q^{-1}[e_{\alpha_i}, e_{\delta-\alpha_i}]_q, \tag{11}$$

$$e_{\alpha_i+n\delta} = (-1)^n \varepsilon_n(\alpha_i)(\overline{\mathrm{ad}}\, e_\delta^{(i)})^n e_{\alpha_i}, \tag{12}$$

$$e_{\delta-\alpha_i+n\delta} = \varepsilon_n(\alpha_i)(\overline{\mathrm{ad}}\, \mathrm{e}_\delta^{(i)})^n e_{\delta-\alpha_i}, \tag{13}$$

$$e_{n\delta}^{\prime(i)} = \varepsilon_n(\alpha_i)[(\alpha_i, \alpha_i)]_q^{-1}[e_{\alpha_i+(n-1)\delta}, e_{\delta-\alpha_i}]_q \tag{14}$$

Here $(\overline{\mathrm{ad}}\, x)y = [x, y]$ is a usual commutator, $\varepsilon_n(\alpha_i) = (-1)^{n\theta(\alpha_i)}$, and θ: $\Pi_0 = \{\alpha_1, \ldots, \alpha_r\} \to \{0, 1\}$ is chosen in such a way that

$$(\alpha_i, \alpha_j) \neq 0 \quad \longrightarrow \quad \theta(\alpha_i) \neq \theta(\alpha_j) \,. \tag{15}$$

Then we use the inductive procedure again to obtain the other real root vectors $e_{\gamma+n\delta}, e_{\delta-\gamma+n\delta}, \gamma \in \Delta_+ \ (g)$. We come to the end by defining the imaginary root vectors $e_{n\delta}^{(i)}$ through the intermediate vectors $e_{n\delta}'^{(i)}$ by means of the following (Schur) relations:

$$e_{n\delta}^{\prime(i)} = \sum_{p_1+2p_2+\ldots+np_n=n} \frac{(q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)})^{\sum p_i - 1}}{p_1! \cdots p_n!} (e_\delta^{(i)})^{p_1} \cdots (e_{n\delta}^{(i)})^{p_n}. \tag{16}$$

In terms of generating functions

$$E_i'(z) = (q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}) \sum_{n\geq 1} e_{n\delta}'^{(i)} z^n$$

and

$$E_i(z) = (q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}) \sum_{n\geq 1} e_{n\delta}^{(i)} z^n$$

relation (16) may be rewritten in an inverted form:

$$E_i(z) = \ln(1 + E_i'(z)). \tag{17}$$

The vectors e_γ and $e_{-\gamma} = (e_\gamma)^*$, $\gamma \in \Delta_+ (\widehat{g})$ are the Cartan–Weyl generators for $U_q(\widehat{g})$.

The following proposition holds [7,8] .

Proposition 3.1. The root vectors $e_{\pm\gamma} \in U_q(g)$, $\gamma \in \Delta_+$, satisfy the relations

$$(e_{\pm\gamma})^* = e_{\mp\gamma}, \quad k_\alpha^{\pm 1} e_\gamma = q^{\pm(\alpha,\gamma)} e_\gamma k_\alpha^{\pm 1}, \tag{18}$$

$$[e_\gamma, e_{-\gamma}] = a(\gamma)\frac{k_\gamma - k_\gamma^{-1}}{q-q^{-1}}, \quad a(\gamma) \in \mathbb{C}, \tag{19}$$

$$[e_\alpha, e_\beta]_q = \sum_{\alpha<\gamma_1<\ldots<\gamma_n<\beta} C_{m_i,\gamma_i} e_{\gamma_1}^{m_1} e_{\gamma_2}^{m_2} \cdots e_{\gamma_n}^{m_n}, \tag{20}$$

where $\sum_i k_i\gamma_i = \alpha + \beta$ and the coefficient C are rational functions of q such that they do not depend on the Cartan elements k_{α_i}, $i = 1,2,\ldots,n$; the monomials (finite products) $e_{\gamma_1}^{n_1} e_{\gamma_2}^{n_2} \cdots e_{\gamma_N}^{n_N}$ $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ and $e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_N}^{n_N}$ $(\gamma_1 < \gamma_2 < \cdots < \gamma_N)$, generate subalgebra $U_q(b_+)$ and $U_q(b_-)$, correspondingly.

Remark. In the relation (18)–(20) the root vectors of imaginary roots $\gamma = \pm n\delta$ have to be labeled by an additional index s: $e_{\pm n\delta}^{(s)}$, $s = 1,2,\ldots,$ mult, where mult is the multiplicity of the imaginary root $\pm n\delta$.

If we introduce the new elements

$$\widetilde{e}_\gamma = e_\gamma, \quad \widetilde{e}_{-\gamma} = -k_\gamma^{-1} e_{-\gamma}, \quad \gamma \in \Delta_+ , \tag{21}$$

then relation (20) is generalized as follows:

$$[\widetilde{e}_{-\beta}, \widetilde{e}_\alpha]_q = \sum C_{m_i,\gamma_i;m_j',\gamma_j'} \widetilde{e}_{-\gamma_1}^{m_1} \widetilde{e}_{-\gamma_2}^{m_2} \cdots \widetilde{e}_{-\gamma_p}^{m_p} \widetilde{e}_{\gamma_1'}^{m_1'} \widetilde{e}_{\gamma_2'}^{m_2'} \cdots \widetilde{e}_{\gamma_s'}^{m_s'}, \tag{22}$$

where the sum is taken on $\gamma_1,\ldots,\gamma_p,\gamma_1',\ldots,\gamma_s'$ and $m_1,\ldots,m_s'$ such that

$$\gamma_1' < \ldots < \gamma_s' < \alpha < \beta < \gamma_1 < \ldots < \gamma_p,$$
$$\sum_l (m_l'\gamma_l' - m_l\gamma_l) = \beta - \alpha \tag{23}$$

and the coefficients C are rational functions of q such that they do not depend on the elements k_{α_i}, $i = 1,2,\ldots,r$.

The imaginary root vectors $e_{n\delta}^{(i)}$ generate the q-analog of the Heisenberg algebra which is described in the following proposition.

Proposition 3.2. The following relations are valid for imaginary root vectors:

$$[e_{\alpha_i+m\delta}, e^{(j)}_{n\delta}] = \frac{q^{n(\alpha_i,\alpha_j)} - q^{-n(\alpha_i,\alpha_j)}}{n(q^{(\alpha_j,\alpha_j)} - q^{-(\alpha_j,\alpha_j)})} e_{\alpha_i+(m+n)\delta}, \quad m \geq 0, n > 0, \tag{24}$$

$$[e^{(j)}_{n\delta}, e_{\delta-\alpha_i+m\delta}] = \frac{q^{n(\alpha_i,\alpha_j)} - q^{-n(\alpha_i,\alpha_j)}}{n(q^{(\alpha_j,\alpha_j)} - q^{-(\alpha_j,\alpha_j)})} e_{\delta-\alpha_i+(m+n)\delta}, \quad m \geq 0, n > 0, \tag{25}$$

$$[e^{(i)}_{n\delta}, e^{(j)}_{m\delta}] = \delta_{m,-n} \frac{(q^{n(\alpha_i,\alpha_j)} - q^{-n(\alpha_i,\alpha_j)})}{n(q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)})(q^{(\alpha_j,\alpha_j)} - q^{-(\alpha_j,\alpha_j)})} (k^n_\delta - k^{-n}_\delta). \tag{26}$$

Remark. Relation (24) is still valid for any integer m and $k > 0$ if we replace $e_{\alpha_i+m\delta}$ by $\widetilde{e}_{\alpha_i+m\delta}$ (see (21)) in both sides. The relation (25) can be extended to negative values of k and m in an analogous manner.

4. Drinfeld's realization of quantum nontwisted affine algebras

V.G. Drinfeld suggested another realization of the quantum nontwisted affine algebra [3]. In this description the algebra $U_q(\widehat{g})$ is generated by the elements c, D, $\xi^\pm_{ik}$, x_{ik} where $i = 1, 2, \ldots, r$; $k \in \mathbb{Z}$ satisfying the relations

$$[c, x_{ik}] = [c, \xi^\pm_{ik}] = [c, D] = 0, \quad [D, x_{ik}] = k\, x_{ik}, \quad [D, \xi_{ik}] = k\, \xi^\pm_{ik}, \tag{27}$$

$$[x_{ik}, x_{jl}] = 4\delta_{k,-l} k^{-1} \hbar^{-2} \sinh(k\hbar B_{ij}) \sinh(k\hbar c/2), \tag{28}$$

$$[x_{ik}, \xi^\pm_{jl}] = \pm 2k^{-1}\hbar^{-1} \sinh(k\hbar B_{ij}) \exp(\mp |k| \cdot \hbar c/4)\, \xi^\pm_{j,k+l}, \tag{29}$$

$$\xi^\pm_{i,k+1}\xi^\pm_{jl} - \mathrm{e}^{\pm\hbar B_{ij}} \xi^\pm_{jl}\xi^\pm_{i,k+1} = \mathrm{e}^{\pm\hbar B_{ij}} \xi^\pm_{ik}\xi^\pm_{j,l+1} - \xi^\pm_{j,l+1}\xi^\pm_{ik} \tag{30}$$

$$[\xi^+_{ik}, \xi^-_{jl}] = \delta_{ij}\hbar^{-1}(\psi_{i,k+l}\, \mathrm{e}^{\hbar c(k-l)/4} - \phi_{i,k+l}\, \mathrm{e}^{\hbar c(l-k)/4}), \tag{31}$$

and q-analogs of the Serre relations, which we do not write down here (see, e.g., ref. [3]). The elements $\phi_{i,p}$, $\psi_{i,p}$ are defined from the relations

$$\sum_p \phi_{i,p} u^{-p} = \exp -\hbar(\tfrac{1}{2}x_{i,0} + \sum_{p<0} x_{i,p} u^{-p}), \tag{32}$$

$$\sum_p \psi_{i,p} u^{-p} = \exp \hbar(\tfrac{1}{2}x_{i,0} + \sum_{p>0} x_{i,p} u^{-p}), \tag{33}$$

$$B_{i,j} = \tfrac{1}{2}(\alpha_i, \alpha_j), \quad i, j = 1, 2, , \ldots, r\,.$$

We should like to show here how to express generators $x_{i,k}$ and $\xi^\pm_{i,k}$ through Cartan–Weyl generators $e_{\alpha_i+n\delta}$, $e_{\delta-\alpha_i+n\delta}$, $e^{(i)}_{\pm n\delta}$. Of course, the resulting relations will differ slightly from (27)–(33), because we use $[n]_q$ instead of $\hbar^{-1}\sinh n\hbar$.

Let $q = \exp(-\frac{1}{2}\hbar)$, $c = h_\delta$. We put

$$x_{i,n} = e^{(i)}_{n\delta}, \quad i = 1, \ldots, r, \quad n \in \mathbb{Z} \tag{34}$$

and

$$\xi^{+}_{i,n} = \widetilde{e}_{\alpha_i+n\delta} q^{-nh_\delta/2}, \quad \xi^{-}_{i,n} = \widetilde{e}^{*}_{-\alpha_i+n\delta} q^{nh_\delta/2}, \quad n \in \mathbb{Z}, \tag{35}$$

$$\psi_{i,n} = (q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}) q^{h_{\alpha_i}} e'^{(i)}_{n\delta}, \quad \phi_{i,n} = (q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}) e'^{(i)}_{-n\delta} q^{-h_{\alpha_i}}, \; n > 0. \tag{36}$$

Then, due to proposition 3.2, we have

Proposition 4.1. The following relations are valid:

$$[x_{ik}, x_{jl}] = \delta_{k,-l} \frac{[k2B_{i,j}]_q}{k[2B_{j,j}]_q[2B_{i,i}]_q} \frac{q^{kc} - q^{-kc}}{q - q^{-1}}, \tag{37}$$

$$[x_{in}, \xi^{\pm}_{jl}] = \mp \frac{[n2B_{i,j}]_q}{n[2B_{j,j}]_q} q^{\pm|n|c/2} \xi^{\pm}_{j,n+l}. \tag{38}$$

Now we give an interpretation of the relations (27)–(33).

(i) In terms of Cartan–Weyl generators the Serre relations are equivalent to the following corollary of proposition 3.1.

$$[e_\alpha, e_\beta]_q = 0,$$

if $\alpha < \beta$ are neighbouring roots in the sense of fixed normal ordering of the root system;

(ii) the defining relations (11)–(14) may be easily generalized to the identities

$$[e_{\alpha_i+n\delta}, e_{\delta-\alpha_i+m\delta}] = C\delta_{i,j} e'^{(i)}_{(n+m+1)\delta}, \tag{39}$$

where C is a constant. Relations (39) rewritten by means of (16) or (17) in terms of the generators $e^{(i)}_{n\delta}$ give us (31).

(iii) Formulae (30) define the commutation relations between the real root vectors $e_{\alpha_i+n\delta}$, $i = 1, \ldots, r$ or $e_{\delta-\alpha_i+m\delta}$, $i = 1, \ldots, r$. Their translation into the language of proposition 3.1 has a more complicated form, except for the simplest case $k = l$, $i = j$ when they mean q-commutation of the neighbouring root vectors again.

5. The Universal *R*-matrix and formulae for comultiplication in $U_q(\widehat{g})$

The universal R-matrix for $U_q(\widehat{g})$ is, by definition, an element of (some extension of) $U_q(\widehat{g}) \otimes U_q(\widehat{g})$, satisfying the following conditions:

$$\Delta'(x) = R\Delta(x)R^{-1} \quad \forall x \in U_q(\widehat{g}) \tag{40}$$

$$(\Delta \otimes \mathrm{id})R = R^{13}R^{23}, \quad (\mathrm{id} \otimes \Delta)R = R^{13}R^{12}. \tag{41}$$

Here Δ' is an opposite comultiplication in $U_q(\widehat{g})$: $\Delta' = \sigma\Delta$, where $\sigma(u \otimes v) = v \otimes u$.

An explicit expression for the universal R-matrix for the quantum nontwisted affine algebras was given in ref. [7]. Namely, for a fixed normal ordering of $\Delta_+(\widehat{g})$, satisfying (9), we can present it as follows:

$$R = R_{\mathrm{re}}^{+} R_{\mathrm{im}} R_{\mathrm{re}}^{-} K. \tag{42}$$

Here $K = q^{\sum_{i,j} d_{ij} h_i \otimes h_j}$, where d_{ij} is an inverse to the (extended) nondegenerate symmetric Cartan matrix $(\widetilde{a}_{ij}^{\mathrm{sym}})$, $i,j = -1, 0, \dots r$,

$$R_{\mathrm{re}}^{+} = \prod_{\gamma\in\Delta_+^{\mathrm{re}},\ \gamma<\delta}^{\rightarrow} R_\gamma, \quad R_{\mathrm{re}}^{-} = \prod_{\gamma\in\Delta_+^{\mathrm{re}},\ \gamma>\delta}^{\rightarrow} R_\gamma, \tag{43}$$

where

$$R_\gamma = \exp_{q_\gamma}\left(a(\gamma)^{-1} e_\gamma \otimes e_{-\gamma}\right), \tag{44}$$

$a(\gamma)$ is the coefficient in the relation (19) of proposition 3.1. and the order in the product (43) coincides with a chosen normal ordering of Δ_+, for the construction of the Cartan–Weyl basis satisfying (9). Finally,

$$R_{\mathrm{im}} = \exp\left(\sum_{n>0} c_{i,j}^{n} e_{n\delta}^{(i)} \otimes e_{-n\delta}^{(j)}\right) \tag{45}$$

where $c_{i,j}^{n}$ is an inverse matrix to

$$\frac{[n2B_{i,j}]_q}{n[2B_{i,i}]_q[2B_{j,j}]_q(q-q^{-1})}.$$

The factors R_γ and R_{im} may be used to describe the comultiplication formulae for nonsimple root vectors analogously to the finite-dimensional case [10,9].

Proposition 5.1. The following identities are valid for any root $\alpha \in \Delta_+$:

$$\Delta'(e_\alpha) = \left(\prod_{\gamma<\alpha} R_\gamma\right)(1\otimes e_\alpha + e_\alpha \otimes q^{-h_\alpha})\left(\prod_{\gamma<\alpha} R_\gamma\right)^{-1}, \tag{46}$$

$$\Delta'(e_{-\alpha}) = \left(\prod_{\gamma<\alpha} R_\gamma\right)(q^{h_\alpha}\otimes e_{-\alpha} + e_{-\alpha} \otimes 1)\left(\prod_{\gamma<\alpha} R_\gamma\right)^{-1}. \tag{47}$$

These formulae, applied to the roots $e_{\pm\alpha_i+n\delta}$, $e_{n\delta}^{(i)}$ give us the formulae for comultiplication of $x_{i,k}$, $\xi_{i,k}^{\pm}$.

In the conclusion we should like to mention that V.G. Drinfeld [3] suggested another comultiplication for quantum affine algebras. In our notation this comultiplication is defined by the convention that

$$\overline{\Delta}(e_{n\delta}^{(i)}) = e_{n\delta}^{(i)} \otimes 1 + q^{-nc} \otimes e_{n\delta}^{(i)}, \quad \overline{\Delta}(e_{-n\delta}^{(i)}) = e_{-n\delta}^{(i)} \otimes q^{nc} + 1 \otimes e_{-n\delta}^{(i)} \tag{48}$$

Analogously to the above statement we state that comultiplication $\overline{\Delta}$ can be obtain from the usual comultiplication (5), (6) by conjugation with R_{real}^{+}.

Note also that the arguments presented in this lecture can be extended to the twisted case (an example for $A_2^{(2)}$ is given in ref. [8]).

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